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STOCHASTIC VISCO-ELASTIC STRAIN MODELED AS A SECOND MOVEMENT WHITE NOISE PROCESS

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Abstract—Current deterministic formulations of models of creep for complicated materials such as concrete have today reached a limit of refinement beyond which further attempts of development are of doubtful value if the randomness of the creep phenomenon is neglected. Some attempts of stochastic modeling of concrete creep are given in the literature. This paper presents a direct further development of these attempts by introducing a stochastic white noise type constitutive equation for uniaxial strain and stress. In the mean the model is consistent with usual linear visco-elasticity theory. The paper demonstrates approximative solution procedures for determining second moment representations for several different types of beam-column problems. In doing this the paper reveals the general difficulties of stochastic process modeling of creep phenomena because the white noise model is likely to represent the simplest possibility of consistent modeling.

The model suggested in this paper is of a general type that may be applicable to several materials. Its motivation originates from a recent discussion about the origin of the observed random scatter of creep measurements on concrete specimens. Therefore the paper specifically uses the word *concrete* for the considered material rather than a more general term.

1. INTRODUCTION

Concrete creep exhibits random scatter to an extent that makes it questionable what is to be gained from application of refined deterministic mathematical formalisms of a complexity beyond what is needed to model the essential gross creep behavior under preservance of mathematical consistency and operational tractability of the model. In consequence of this insight several papers focus on the question of modeling the uncertainty of creep in probabilistic terms, see e.g. [1–3].

There is a discussion going on whether part of the random scatter is due to the fact that the micro structure creep phenomenon may be viewed as a random process in time and space. Jordaan in [3] and in a discussion with Cinlar, *et al.* [4] claims that the additive effect of a very large number of microscopic events just as in mechanical statistical gas theory shows up as a deterministic macroscopic phenomenon. In his argument he refers to the fact that the standard deviation of an average of n uncorrelated random variables with identical standard deviations σ decreases inversely proportional to $\sqrt{n}$. Furthermore he refers to the central limit theorem to argue that any macroscopic effect of creep will be normally distributed with a very small standard deviation effectively making the phenomenon deterministic. The point where this last conclusion fails is, however, that also σ may be very large such that $\sigma/\sqrt{n}$ is not vanishing even in the macroscopic scale. To appreciate this, the following elementary demonstration is useful. Let $S(x)$ be a random "micro structure" strain process along the x-axis. Then the macroscopic strain over the length L may for any n be decomposed as follows:

$$\frac{1}{L}\int_0^L S(x)\,dx = \frac{1}{L}\left(\int_0^{L/n} + \int_{L/n}^{2L/n} + \cdots + \int_{(n-1)L/n}^{L}\right) S(x)\,dx \tag{1.1}$$

where the integrals may be interpreted in the mean square sense. Assuming that the integrals on the r.h.s. of (1.1) are uncorrelated and all have the same variance the following variance relation results:

$$\mathrm{Var}\left[\frac{1}{L}\int_0^L S(x)\,dx\right] = \frac{1}{n}\mathrm{Var}\left[\frac{n}{L}\int_0^{L/n} S(x)\,dx\right] \tag{1.2}$$

where the variance entering the r.h.s. is the variance of the macroscopic strain over the length L/n. Thus this variance increases proportional to n. For large n the assumption of no correlation between the integrals in (1.1) may be less realistic. Suitable introduction of correlation prevents, however, that the macroscopic strain over the length L/n gets a variance that approaches infinity as $n \to \infty$. On the other hand, the variance of the macroscopic strain over the length L is for large L asymptotically equal to the variance calculated under the assumption that there is no correlation between $S(x)$ and $S(y)$ for any $x \neq y$ and that $S(x)$ has infinite variance in the sense that

$$\mathrm{Cov}[S(x), S(y)] = \gamma\delta(x-y). \tag{1.3}$$

Here $\delta(\cdot)$ is the Dirac delta function and γ is a positive constant which is adjusted to give asymptotically the correct variance of the macroscopic strain over the length L for $L \to \infty$. A formal process of this artificial but operationally very useful type is called a "white noise" process (or "rain on the roof" process) of intensity γ and it is widely used in several engineering fields such as random vibration theory, signal detection theory, systems with random inputs, etc. To see how γ is adjusted let $S(x)$ be a stationary strain process with covariance function $c(x-y) = \mathrm{Cov}[S(x), S(y)]$. Then

$$\mathrm{Var}\left[\frac{1}{L}\int_0^L S(x)\,dx\right] = \frac{1}{L^2}\int_0^L\int_0^L c(x-y)\,dx\,dy = \frac{2}{L}\int_0^L c(u)\,du - \frac{2}{L^2}\int_0^L uc(u)\,du. \tag{1.4}$$

In particular the assumption of $S(x)$ being a white noise process of intensity γ yields

$$\mathrm{Var}\left[\frac{1}{L}\int_0^L S(x)\,dx\right] = \frac{\gamma}{L}. \tag{1.5}$$

Uñder the assumption that $\int_0^\infty uc(u)\,du < \infty$ we then get asymptotically equal results if we put

$$\gamma = 2\int_0^\infty c(u)\,du. \tag{1.6}$$

With reference to the micro structure scale of the cement paste in concrete "effective infinity ∞" is indeed not very large compared to the dimensions of the concrete specimen. Thus some type of white noise strain process assumption may produce useful results.

The reader should note that as a *formal* tool in *second moment uncertainty analysis* of the type considered herein, a white noise process model imposes only extremely weak conditions on the individual creep samples. Specifically it should be noted that the white noise model is not in conflict with the assumption that "basic creep" strain cannot decrease under constant or increasing stress because there exist processes with *nonnegative* sample functions and which have covariance properties that in integration contexts may be approximated by the properties of the Dirac delta function concept. Obviously, integrals of such processes have nondecreasing sample functions of upper integration limit. In order to emphasize that the modeling herein is not dependent on a choice of a specific process having nonnegative sample functions, the name "second moment white noise process" may be used. (Alternative: "wide sense white noise process".)

Cinlar *et al.* [2] give detailed argumentations for assuming a concrete creep strain model that produces so-called infinitely divisible probability distributions of macroscopic strain quantities and they argue for adopting gamma-type distributions.† In this paper focus is solely on mean, variance and covariances. With respect to these second moment uncertainty measures the intuitive arguments of [2] may all be taken as arguments for adopting a second moment white noise spatial strain process model.

Benjamin *et al.* [1] essentially state the same arguments leading to the infinite divisibility assumption. Even though their purpose is to calculate solely second moment representations they do not directly take the step to model the spatial behavior of creep by a second moment white noise process assumption.

Section 2 of this paper presents an integral type constitutive equation, (2.1), that for given uniaxial stress history (to be selected as a free independent variable) defines the uniaxial strain

†This distribution model is not, as claimed by Jordaan [3] in conflict with the central limit theorem since asymptotically these gamma-distributions become normal for $L \to \infty$.

history in terms of some random process defined up to second moment properties, at least. Neglecting compatibility conditions, the consequences are investigated of the equations of equilibrium and this constitutive equation using a second moment white noise integrand. The examples are a cylindrical column and, in Section 3, a beam-column. Section 4 discusses the effect of neglecting micro-geometrical compatibility conditions. Section 4 also considers relaxation type problems, i.e. problems where macro-geometrical compatibility conditions are introduced. These cause coupling between local strains and stresses such that also the stress field history becomes a random process. In fact, the problem is the extremely difficult one of inverting the stochastic constitutive equation (2.1). Presumably it is not solvable by analytical methods. However, by a suitable linearization type of approximation it turns out to be possible to reformulate the equation in such a way that integral equations are readily formulated for means and covariances related to the stress history.

The diversities of opinion whether part of the random scatter of creep quantities is due to macroscopic appearance of microstructure random behavior are, perhaps, not very important from the point of view of studying the random behavior of creep. This becomes clear by reference to an elementary but very useful theorem of second moment analysis which the author in [5] named "the total representation theorem". Let $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_n)$ be "external" random parameters that influence the creep. Some of these may be fixed for a given specimen. These are cement-water ratio, aggregate parameters, cement property parameters, etc. that vary randomly from batch to batch but which are fixed at more or less unknown values the same moment as the production of the specimen is completed. Others may represent the environmental conditions of the specimen during its lifetime. These are temperature, humidity, loading, etc. and they may be modeled as random processes developing in time and space. Let $X(t)$ be any creep effect developing in time, e.g. the midspan deflection of a simply supported beam. Then the "internal" random behavior of the creep may be modeled in terms of the conditional second moment representation

$$E[X(t)|\boldsymbol{\theta}], \quad t \in R_0 \tag{1.7}$$

$$\mathrm{Cov}[X(s), X(t)|\boldsymbol{\theta}], \quad s, t \in R_0. \tag{1.8}$$

According to Jordaan's point of view $X(t)$ for given $\boldsymbol{\theta}$ is a deterministic function of time which is identical then to the conditional mean $E[X(t)|\boldsymbol{\theta}]$ and $\mathrm{Cov}[X(s), X(t)|\boldsymbol{\theta}] = 0$ for all $s, t \in R_0$. For the purpose of second moment analysis it does no harm, however, to keep this question open since according to the total representation theorem [5] we have

$$E[X(t)] = E[E[X(t)|\boldsymbol{\theta}]] \tag{1.9}$$

$$\mathrm{Cov}[X(s), X(t)] = \mathrm{Cov}[E[X(s)|\boldsymbol{\theta}], E[X(t)|\boldsymbol{\theta}]] + E[\mathrm{Cov}[X(s), X(t)|\boldsymbol{\theta}]]. \tag{1.10}$$

It is seen that the assumption that no "internal" random scatter is present on the macroscopic level only implies that the second term of the covariance formula (1.10) is neglected as unimportant compared to the first term. Jordaan [3] has performed some few measurements that speak in favour of this point of view. With the very few experimental results available it cannot be decisively concluded, however, that this will always be so irrespective of the type of concrete. In any case, if the last term in (1.10) is minor or small compared to the first term the difficulties of calculating $\mathrm{Cov}[X(s), X(t)|\boldsymbol{\theta}]$ without simplifying assumptions get less importance for the total result. Thus even rather crude evaluations of $\mathrm{Cov}[X(s), X(t)|\boldsymbol{\theta}]$ may be sufficient for engineering purposes. On the other hand, approximative computer calculations of the mean (1.9) and the first term in (1.10) causes no fundamental difficulties provided a linear viscoelastic constitutive law is adopted. Elegant numerical tools for calculating the mean are developed by Madsen [6] and the author [5] has described a second moment analysis matrix formalism for uncertain systems which is applicable in connection with Madsen's method to calculate the first term of (1.10). Results of this kind of calculations are reported in [7].

2. DEFINITION OF THE SECOND MOMENT WHITE NOISE CREEP MODEL

The modeling is restricted to formulation of a uniaxial strain type model corresponding to ordinary technical beam-column theory in which shear force deformations are neglected. The x axis is in beam direction while y and z are cross-section coordinates. The vector r defines the point (x, y, z). At r there is a given "normal-to-cross-section" stress history $\sigma(r, t)$ as function of time t with $\sigma(r, t) = 0$ for $t \le 0$.

The concrete creep strain in direction of the x-axis at r is modeled by the formal integral (see the remark following (2.6)):

$$\epsilon(r, t) = \int_{\tau=0}^{t} \int_{\sigma(r,\tau)}^{\sigma(r,\tau)+d\sigma(r,\tau)} S(r, t, \tau, u)\, du \tag{2.1}$$

where $S(r, t, \tau, u)$ is a second moment white noise process in geometrical (r), time (τ), stress (u) space such that

$$E[S(r, t, \tau, u)] = K(t, \tau) \tag{2.2}$$

is a function solely of t and τ while

$$\mathrm{Cov}[S(r_1, t_1, \tau_1, u_1), S(r_2, t_2, \tau_2, u_2)] = c(t_1, t_2, \tau_1)\delta(\tau_2 - \tau_1)\delta(u_2 - u_1)\delta(r_2 - r_1) \tag{2.3}$$

where $\delta(\cdot)$ is the Dirac delta function, $\delta(r_2 - r_1) = \delta(x_2 - x_1)\, \delta(y_2 - y_1)\, \delta(z_2 - z_1)$ and $c(t_1, t_2, \tau_1)$ is a suitable function that will be interpreted below.

The integral (2.1) should, naturally, be interpreted in the mean square sense. The mean strain at r becomes

$$E[\epsilon(r, t)] = \int_{\tau=0}^{t} \int_{\sigma(r,\tau)}^{\sigma(r,\tau)+d\sigma(r,\tau)} E[S(r, t, \tau, u)]\, du = \int_{\tau=0}^{t} K(t, \tau)\, d\sigma(r, \tau) \tag{2.4}$$

which is the usual creep integral, the function $K(t, \tau)$ being the creep function of usual linear visco-elasticity theory.

For the covariance we get

$$\begin{aligned}
&\mathrm{Cov}\left[\int_0^{t_1} \int_{\sigma(r_1,\tau_1)}^{\sigma(r_1,\tau_1)+d\sigma(r_1,\tau_1)} S(r_1, t_1, \tau_1, u_1)\, du_1,\ \int_0^{t_2} \int_{\sigma(r_2,\tau_2)}^{\sigma(r_2,\tau_2)+d\sigma(r_2,\tau_2)} S(r_2, t_2, \tau_2, u_2)\, du_2\right] \\
&= \delta(r_2 - r_1) \int_0^{t_1} c(t_1, t_2, \tau_1) \int_0^{t_2} \delta(\tau_2 - \tau_1) \int_{\sigma(r_1,\tau_1)}^{\sigma(r_1,\tau_1)+d\sigma(r_1,\tau_1)} du_1 \int_{\sigma(r_2,\tau_2)}^{\sigma(r_2,\tau_2)+d\sigma(r_2,\tau_2)} \delta(u_2 - u_1)\, du_2 \\
&= \delta(r_2 - r_1) \int_0^{t_1} c(t_1, t_2, \tau_1) \int_0^{t_2} \delta(\tau_2 - \tau_1) \int_{\sigma(r_1,\tau_1)}^{\sigma(r_1,\tau_1)+d\sigma(r_1,\tau)} 1_{[\sigma(r_2,\tau_2),\, \sigma(r_2,\tau_2)+d\sigma(r_2,\tau_2)]}(u_1) \frac{d\sigma(r_2, \tau_2)}{|d\sigma(r_2, \tau)|}\, du_1 \\
&= \delta(r_2 - r_1) \int_0^{t_1} c(t_1, t_2, \tau_1) \int_0^{t_2} \delta(\tau_2 - \tau_1) |d\sigma(r_1, \tau_1)| \\
&= \delta(r_2 - r_1) \int_0^{t_1} c(t_1, t_2, \tau_1) 1_{[0,t_2]}(\tau_1) |d\sigma(r_1, \tau_1)|
\end{aligned} \tag{2.5}$$

in which symbols of type $1_A(x)$ means indicator function of set A, that is, $1_A(x) = 1$ for $x \in A$, $1_A(x) = 0$ for $x \notin A$. Thus the covariance is

$$\mathrm{Cov}[\epsilon(r_1, t_1), \epsilon(r_2, t_2)] = \delta(r_2 - r_1) \int_0^{t_1} c(t_1, t_2, \tau) |d\sigma(r_1, \tau)| \tag{2.6}$$

for $t_1 \le t_2$. The step from the third to the fourth expression of (2.5) consists in changing (r_2, τ_2) to (r_1, τ_1) and integrating. This is justified because of the delta function factors $\delta(r_2 - r_1)$ $\delta(\tau_2 - \tau_1)$.

Remark: If the model (2.1) is changed to

$$\epsilon(r,t)=\int_{\tau=0}^{t}S(r,t,\tau)\,\mathrm{d}\sigma(r,\tau) \tag{2.1a}$$

with

$$E[S(r,t,\tau)]=K(t,\tau),\mathrm{Cov}[S(r_1,t_1,\tau_1),S(r_2,t_2,\tau_2)]=c(t_1,t_2,\tau_1)\,\delta(\tau_2-\tau_1)\,\delta(r_2-r_1)$$

then (2.4) still results. The covariance, however, does not take the form of (2.6). It becomes

$$\mathrm{Cov}[\epsilon(r_1,t_1),\epsilon(r_2,t_2)]=\delta(r_2-r_1)\int_{\tau_1=0}^{t_1}c(t_1,t_2,\tau_1)\,\mathrm{d}\sigma(r_1,\tau_1)\,\mathrm{d}\sigma(r_2,\tau_1)$$

showing integration with respect to the squared stress increment while (2.5) shows integration with respect to the absolute value of the stress increment.

This property of (2.6) is, in fact, crucial for the possibility of deducing to a certain extent from stress level results to internal force results as done in this paper.

The arguments of [2] are pointing at the model (2.1) rather than the seemingly simplified model (2.1a) which follow from (2.1) by improper application of the mean value theorem of integral theory.

Let us consider a cylindrical column $\mathcal{B}$ of length L and cross-section ω with area A subjected to a normal force $N(\tau)$ as function of time $\tau\geq 0$. The average strain over the column then becomes

$$\epsilon(t)=\frac{1}{L}\int_0^L \mathrm{d}x\,\frac{1}{A}\int_\omega \epsilon(r,t)\,\mathrm{d}y\,\mathrm{d}z \tag{2.7}$$

with expectation

$$E[\epsilon(t)]=\frac{1}{L}\int_0^L \mathrm{d}x\frac{1}{A}\int_\omega \mathrm{d}y\,\mathrm{d}z\int_{\tau=0}^{t}K(t,\tau)\,\mathrm{d}\sigma(r,\tau)=\frac{1}{L}\int_0^L\mathrm{d}x\int_{\tau=0}^{t}K(t,\tau)\frac{1}{A}\int_\omega \mathrm{d}\sigma(x,y,z,\tau)$$
$$=\int_{\tau=0}^{t}K(t,\tau)\frac{\mathrm{d}N(\tau)}{A}. \tag{2.8}$$

The covariance function becomes for $t_2\geq t_1$:

$$\mathrm{Cov}[\epsilon(t_1),\epsilon(t_2)]=\frac{1}{(LA)^2}\int_0^L\mathrm{d}x_1\int_0^L\mathrm{d}x_2\int_\omega\mathrm{d}y_1\,\mathrm{d}z_1\int_\omega\mathrm{Cov}[\epsilon(r_1,t_1),\epsilon(r_2,t_2)]\,\mathrm{d}y_2\,\mathrm{d}z_2$$
$$=\frac{1}{(LA)^2}\int_0^{t_1}c(t_1,t_2,\tau)\int_0^L\mathrm{d}x_1\int_0^L\delta(x_2-x_1)\,\mathrm{d}x_2\int_\omega|\mathrm{d}\sigma(r_1,\tau)|\,\mathrm{d}y_1\,\mathrm{d}z_1$$
$$\times\int_\omega\delta(y_2-y_1)\,\delta(z_2-z_1)\,\mathrm{d}y_2\,\mathrm{d}z_2=\frac{1}{(LA)^2}\int_0^{t_1}c(t_1,t_2,\tau)\int_{\mathcal{B}}|\mathrm{d}\sigma(r,\tau)|$$
$$\geq\frac{1}{LA}\int_0^{t_1}c(t_1,t_2,\tau)\frac{|\mathrm{d}N(\tau)|}{A}. \tag{2.9}$$

With a unit average normal stress applied to time T and kept constant for $t\geq T$ we have $\mathrm{d}N(\tau)=A\delta(\tau-T)\,\mathrm{d}\tau$ and thus

$$E[\epsilon(t)]=K(t,T),\ t\geq t_0 \tag{2.10}$$

$$\mathrm{Cov}[\epsilon(t_1),\epsilon(t_2)]\geq\frac{1}{LA}c(t_1,t_2,T),\quad t_2\geq t_1\geq T. \tag{2.11}$$

It is emphasized that the results (2.4)–(2.11) are all conditional on a given stress field history.

Based on commonly suggested micro mechanisms of concrete creep such as they are reflected in deterministic creep theories Benjamin *et al.* [1] assume that there essentially are two independent stochastic creep components, viscous flow and delayed elasticity. These authors model the viscous flow as a process with independent increments and even more specifically they suggest, except for proportionality with constant C_V, a nonhomogeneous Poisson process with (as an example) mean rate γ/t for $t \geq T$. Then the viscous part of the creep function gets the form

$$K_1(t, T) = C_V\gamma \log \frac{t}{T}, \; t \geq T \tag{2.12}$$

while the corresponding part of $c(t_1, t_2, T)$ becomes

$$c_1(t_1, t_2, T) = C_V^2\gamma \log \frac{t_1}{T} \text{ for } t_2 \geq t_1 \geq T. \tag{2.13}$$

For the delayed elasticity part Benjamin *et al.* [1] suggest a Markov birth process, an assumption that they also motivate by micromechanism considerations. As an example they consider a time-homogeneous birth process implying that the delayed elasticity part of the creep function gets the form

$$K_2(t, T) = \frac{1}{E(T)}(1 - e^{-\alpha(t-T)}), \quad t \geq T \tag{2.14}$$

where α is a positive constant and $E(T)$ is the delayed elasticity modulus at the age of loading T. The corresponding part of $c(t_1, t_2, T)$ becomes

$$c_2(t_1, t_2, T) = \frac{C_E}{E(T)} e^{-\alpha(t_2-T)}(1 - e^{-\alpha(t_1-T)}), \quad t_2 \geq t_1 \geq T \tag{2.15}$$

where C_E is a positive constant. Both α and C_E have specific interpretations in [1]. The creep function is then $K = K_1 + K_2$ while $c = c_1 + c_2$. By design the final creep function K coincides in form with the (deterministic) creep function suggested by Hansen [8].

Clearly several alternatives are possible of models which are formulated by use of processes with independent increments (the viscous part) and Markov birth processes (the delayed elasticity part). Cinlar *et al.* [2] also argue in favour of modeling the viscous part by a process with independent increments. However, instead of using a nonhomogeneous Poisson process they recommend a gamma-type process. This has, however, no strong influence on the modeling of the second moment representation of the creep. Essential features of its behavior are determined by the mere assumption of independent increments and the direct choice of the mean value function to fit some given deterministic creep function.

The purpose of the discussion herein is not to take favour of one or the other model but solely to demonstrate that the second moment white noise model defined by (2.1)–(2.3) has enough generality to be adapted to any such model leading to explicit expressions for the creep function $K(t, \tau)$ and the covariance function $c(t_1, t_2, \tau)$.

3. COMBINED COMPRESSION AND BENDING OF BEAM-COLUMN OF PLAIN CONCRETE

A cylindric beam column with symmetrical cross-section ω is subjected to a normal compression force $N(x, t)$ and a bending moment $M(x, t)$ causing bending in the symmetry plane such as indicated in Fig. 1. Both $N(x, t)$ and $M(x, t)$ are assumed to be given functions of axial coordinate x and time t. The entire section carries stresses, that is, no cracking takes place. In order to formulate a stochastic creep bending model let us define processes $B(x, t)$ and $C(x, t)$ developing along x-axis and in time in such a way that the integral

$$\int_\omega (B(x, t) - yC(x, t) - \epsilon(x, y, z, t))^2 \tag{3.1}$$

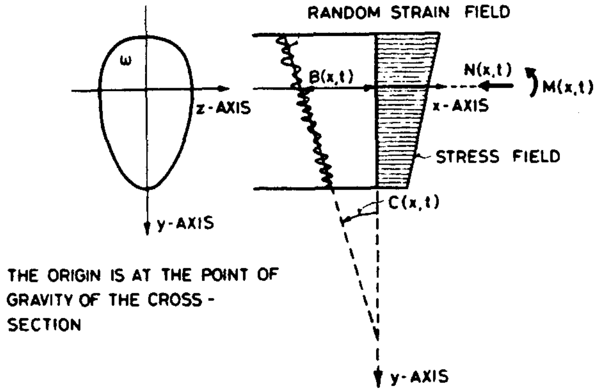

Fig. 1. Definition sketch of cross-section in beam of plain concrete subjected to compression and bending in symmetry plane.

is minimized. From setting the partial derivatives of (3.1) with respect to B and C to zero we get the definition

$$B(x, t) = \frac{1}{A}\int_\omega \epsilon(x, y, z, t) \tag{3.2}$$

$$C(x, t) = -\frac{1}{I}\int_\omega y\epsilon(x, y, z, t) \tag{3.3}$$

in which A and I are area and moment of inertia (with respect to z-axis) respectively of the cross-section ω. To constitute the model we set up the generalized Bernoulli type model: *The curvature of the beam axis at x to time t is given by $C(x, t)$ (Bernoulli modeling hypothesis* 1).

Let $\sigma(r, \tau)$, $\tau \leq t$, be a stress field history which in any cross-section is statically equivalent with the given pair of internal forces in that cross-section. By use of (2.4) we then get

$$E[B(x, t)] = \frac{1}{A}\int_{\tau=0}^{t} K(t, \tau)\int_\omega \mathrm{d}\sigma(x, y, z, \tau) = \frac{1}{A}\int_{\tau=0}^{t} K(t, \tau)\,\mathrm{d}N(x, \tau) \tag{3.4}$$

$$E[C(x, t)] = -\frac{1}{I}\int_{\tau=0}^{t} K(t, \tau)\int_\omega y\,\mathrm{d}\sigma(x, y, z, \tau) = \frac{1}{I}\int_{\tau=0}^{t} K(t, \tau)\,\mathrm{d}M(x, \tau). \tag{3.5}$$

Let the cross-sections at x_1 and x_2 be ω_1 and ω_2 respectively. By use of (2.6) we then get the covariance

$$\begin{aligned}\mathrm{Cov}\left[\int_{\omega_1} y_1\epsilon(r_1, t_1), \int_{\omega_2} y_2\epsilon(r_2, t_2)\right] &= \int_{\omega_1}\int_{\omega_2} y_1y_2\delta(r_2 - r_1)\int_0^{t_1} c(t_1, t_2, \tau)|\mathrm{d}\sigma(r_1, \tau)| \\ &= \delta(x_2 - x_1)\int_0^{t_1} c(t_1, t_2, \tau)\int_{\omega_1} y_1^2|\mathrm{d}\sigma(x_1, y_1, z_1, \tau)|\end{aligned}$$

which by comparison with (3.3) shows that

$$\mathrm{Cov}[C(x_1, t_1), C(x_2, t_2)] = \delta(x_2 - x_1)\frac{1}{I^2}\int_0^{t_1} c(t_1, t_2, \tau)\int_\omega y^2|\mathrm{d}\sigma(x_1, y, z, \tau)| \tag{3.6}$$

for $t_1 \leq t_2$.

In order to transfer from the stress level of modeling to the internal force level of modeling it is necessary at this place to make a plausible choice of an equilibrium stress field. The simplest choice is to take a stress field $\sigma(x, y, z, t)$ *that in any cross-section varies linearly with y and z (Bernoulli modeling hypothesis* 2).

For symmetrical bending this hypothesis implies that

$$\sigma(x, y, z, t) = \frac{1}{A}N(x, t) - \frac{y}{I}M(x, t) \tag{3.7}$$

such that (3.6) gives

$$\mathrm{Cov}[C(x_1,t_1),\,C(x_2,t_2)]=\delta(x_2-x_1)\frac{1}{I^2}\int_0^{t_1}c(t_1,t_2,\tau)\int_\omega y^2\left|\frac{1}{A}\,\mathrm{d}N(x_1,\tau)-\frac{y}{I}\,\mathrm{d}M(x_1,\tau)\right|. \tag{3.8}$$

The curvature process $C(x,t)$ is a second moment white noise process in x. By the virtual work principle we get the deflection process

$$U(x,t)=-\int_0^L G(\xi,x)C(\xi,t)\,\mathrm{d}\xi \tag{3.9}$$

in which L is the total length of the beam-column and $G(\xi,x)$ is the bending moment at the place ξ for a unit force acting at x in direction of the deflection. This last integration "smoothen" the white noise process. We get

$$E[U(x,t)]=-\int_0^L G(\xi,x)E[C(\xi,t)]\,\mathrm{d}\xi=\frac{1}{I}\int_{\tau=0}^{t}K(t,\tau)\,\mathrm{d}\Big(\int_0^L G(\xi,x)M(\xi,\tau)\,\mathrm{d}\xi\Big) \tag{3.10}$$

$$\begin{aligned}\mathrm{Cov}[U(x_1,t_1),\,U(x_2,t_2)]&=\int_0^L\int_0^L G(\xi_1,x_1)G(\xi_2,x_2)\,\mathrm{Cov}[C(\xi_1,t_1),\,C(\xi_2,t_2)]\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2=\\&=\int_0^{t_1}c(t_1,t_2,\tau)\frac{1}{I^2}\int_0^L G(\xi,x_1)G(\xi,x_2)\Big(\int_\omega y^2\left|\frac{1}{A}\,\mathrm{d}N(\xi,\tau)-\frac{y}{I}\,\mathrm{d}M(\xi,\tau)\right|\Big)\,\mathrm{d}\xi\end{aligned} \tag{3.11}$$

for $t_1\le t_2$.

Example 1

Consider a cantilever beam subjected to a constant compression force N at the free end for $t\ge T$. Bending moments from N due to deflections are neglected and no external bending moment is applied. Thus the mean deflection perpendicular to the beam is zero. From (3.11) we get for $x_1\le x_2$ and $T\le t_1\le t_2$:

$$\begin{aligned}\mathrm{Cov}[U(x_1,t_1),\,U(x_2,t_2)]&=c(t_1,t_2,T)\frac{N}{AI}\int_0^{x_1}(x_1-\xi)(x_2-\xi)\,\mathrm{d}\xi\\&=\frac{1}{6}x_1^2(3x_2-x_1)\frac{N}{AI}c(t_1,t_2,T)\end{aligned} \tag{3.12}$$

giving the variance

$$\mathrm{Var}[U(x,t)]=\frac{1}{3}x^3\frac{N}{AI}c(t,t,T) \tag{3.13}$$

and the correlation coefficient

$$\rho[U(x_1,t),\,U(x_2,t)]=\frac{1}{2}\sqrt{\frac{x_1}{x_2}}\Big(3-\frac{x_1}{x_2}\Big). \tag{3.14}$$

For $x_1=(1/2)L$ and $x_2=L$ the right side is $5/4\sqrt{2}\simeq 0.88$. This indicates a pronounced tendency of simultaneous deflection to the same side of the underformed beam-column in spite of what would have been naively expected, perhaps, from the white noise character of the curvature process.

Example 2

In order to be able to analyse the creep deflection behavior of a reinforced concrete beam by the method of this section it is necessary to make some further idealizing assumptions besides the two Bernoulli hypotheses. These may, for example, for a cross-section as indicated

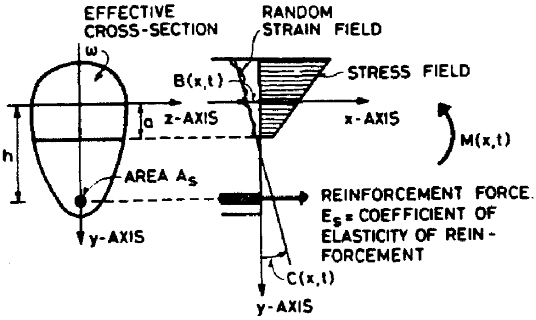

Fig. 2. Definition sketch of reinforced cross-section subjected to bending in symmetry plane.

in the definition sketch Fig. 2 be as follows:

(1) Solely that part of the concrete section for which $y \le a$ (some suitably selected constant) carries longitudinal stresses (the *effective* section assumption).

(2) The reinforcement force $N(x, t)$ is *nonrandom* and determined by a strain of size $E[B(x, t) - hC(x, t)]$ in the reinforcement. Here h is the distance between the points of gravity of the effective concrete section and the reinforcement section.

The results (3.4) and (3.5) apply to give

$$\begin{aligned} N(x,t) &= E_sA_s(hE[C(x,t)] - E[B(x,t)]) \\ &= E_sA_s \int_{\tau=0}^{t} K(t,\tau)\,\mathrm{d}\Big(\frac{h}{I}(M(x,\tau) - hN(x,\tau)) - \frac{1}{A}N(x,\tau)\Big) = \\ &= -(h^2A + I)\frac{E_sA_s}{IA}\int_{\tau=0}^{t} K(t,\tau)\,\mathrm{d}N(x,\tau) + \frac{hE_sA_s}{I}\int_{\tau=0}^{t} K(t,\tau)\,\mathrm{d}M(x,\tau) \end{aligned} \tag{3.15}$$

This is an inhomogeneous integral equation from which $N(x, t)$ may be determined when given the bending moment history $M(x, t)$. For example, if the bending moment is applied to time $t = 0$ and kept constant eqn (3.15) becomes

$$N(x,t) = -c_1\int_{\tau=0}^{t} K(t,\tau)\,\mathrm{d}N(x,\tau) + c_2K(t,0)M(x,0+) \tag{3.16}$$

where c_1, c_2 are the section constants of (3.15). A variety of numerical techniques are available to solve (3.15) or (3.16) for each value of x. For some very special choices of the creep function $K(t, \tau)$ close form solutions may be worked out, for example by use of Laplace transform methods.

With $N(x, t)$ determined from (3.15) the covariance function of the curvature next follows from formula (3.8).

The choice of effective concrete section is arbitrary but a reasonable choice of the distance a is, perhaps, defined by the instantaneous position of the plane of zero strain calculated under the assumption that the concrete is unable to carry longitudinal tensile stresses.

In order to avoid the somewhat dubious assumption 2 it is necessary to linearize the strain integral (2.1) in a way as illustrated in next section. It is then possible to carry through the analysis by assuming that $N(x, t)$ is the random process

$$N(x,t) = E_sA_s(hC(x,t) - B(x,t)). \tag{3.17}$$

The mean $E[N(x, t)]$ will still be the solution to the integral equation (3.15) while the determination of the covariance functions of $N(x, t)$ and $C(x, t)$ become much more laborious. Next section discusses for illustration some considerably simpler problems of relaxation type. The solution methodology of these problems may, however, be extended to the reinforced beam problem. Comparisons of the results of this kind of analysis and the results based on the strongly simplifying assumption 2 are reported in [7]. It turns out that the uncertainty increases considerably by letting $N(x, t)$ be defined as the random process (3.17) instead of letting $N(x, t)$ be the mean (3.15) of the process. The simply supported beam example of [7] shows about a doubling of the standard deviation of the midspan deflection.

4. DIFFICULTIES OF THE PROPOSED MODEL: THE NEGLIGENCE OF MICROGEOMETRICAL COMPATIBILITY CONDITIONS. RELAXATION TYPE PROBLEMS

Both the consequences (2.8), (2.9) and those of Section 3 of the stochastic constitutive equation (2.1) were based on the assumption that the entire equilibrium stress field history can be selected completely independent of the strain field history. This is plausible only when no micro-geometrical compatibility conditions are stated. Such compatibility conditions will generally have a tendency to "smoothen" the random spatial variation of the strain field in such a way that smaller variances result.

On the other hand, as explained in the introduction the second moment white noise strain model is a mathematical idealization that serves to give asymptotically correct results after integration over "large" bodies. Therefore it is possible to work with a stress definition where stress is defined as an average of internal forces over suitably large area-elements of the cross-section making the stress almost independent of micro-geometrical compatibility conditions. The Bernoully modeling hypothesis 2 leading from (3.6) to (3.8) is based on this kind of stress definition. Accepting this hypothesis for the stress field in (2.9) implies that the inequalities of (2.9) and (2.11) become equalities.

Severe mathematical difficulties appear in connection with any problem of type like the relaxation problem: Assume that a given total strain history $\epsilon(t)$ (with $\epsilon(t)=0$ for $t\leq 0$) is forced upon the column $\mathcal{B}$ of Section 2. The problem is to determine the second moment representation of the normal force process $N(t)$ for $t\geq 0$ as it is generated due to the stochastic constitutive equation (2.1).

With the instantaneous elastic strain included in the strain process $S(r, t, \tau, u)$ the condition that the random stress field history $\sigma(r, t)$ must satisfy is

$$\int_{\tau=0}^{t}\int_{\mathcal{B}}\int_{\sigma(r,\tau)}^{\sigma(r,\tau)+d\sigma(r,\tau)} S(r, t, \tau, u) = LA\epsilon(t). \tag{4.1}$$

To find the second moment properties of

$$N(t)=\frac{1}{L}\int_{\mathcal{B}}\sigma(r, t) \tag{4.2}$$

from this equation is a very hard problem that most likely is not practicable by known mathematical methods. If, however, the approximation

$$\int_{\sigma(r,\tau)}^{\sigma(r,\tau)+d\sigma(r,\tau)} S(r, t, \tau, u) = K(t,\tau)(d\sigma(r,\tau)-dE[\sigma(r,\tau)])+ + \int_{E[\sigma(r,\tau)]}^{E[\sigma(r,\tau)]+dE[\sigma(r,\tau)]} S(r, t, \tau, u) \tag{4.3}$$

is substituted into the l.h.s. of (4.1) a more tractable equation appears. It is

$$\int_{\tau=0}^{t}\int_{\mathcal{B}}\int_{E[\sigma(r,\tau)]}^{E[\sigma(r,\tau)]+dE[\sigma(r,\tau)]} S(r, t, \tau, u) = LA\epsilon(t)+L\int_{\tau=0}^{t} K(t,\tau)(dE[N(\tau)]-dN(\tau)). \tag{4.4}$$

Taking expectation we get the integral equation

$$\int_{\tau=0}^{t} K(t,\tau)\, dE[N(\tau)] = A\epsilon(t) \tag{4.5}$$

from which the expectation $E[N(t)]$ may be determined by standard methods.

The covariance function of the l.h.s. of (4.4) is given by, see (2.9),

$$\int_0^{t_1} c(t_1, t_2, \tau)\int_{\mathcal{B}}|dE[\sigma(r,\tau)]| = L\int_0^{t_1} c(t_1, t_2, \tau)|dE[N(\tau)]|,\ t_2\geq t_1 \tag{4.6}$$

where " = " applies and not just " ≥ " (compare with (2.9)) because the stochastic homogeneity of the strain process causes $dE[\sigma(r, \tau)]$ to have the same sign everywhere in $\mathscr{B}$. By use of the solution to (4.5) this covariance function becomes a known function $F(t_1, t_2)$. It is equal to the covariance function of the r.h.s. of (4.4). The result is the integral equation

$$L^2\int_{\tau_1=0}^{t_1}\int_{\tau_2=0}^{t_2} K(t_1, \tau_1)K(t_2, \tau_2)\,\mathrm{Cov}[\mathrm{d}N(\tau_1), \mathrm{d}N(\tau_2)] = F(t_1, t_2). \tag{4.7}$$

This equation may be solved by numerical standard technique that approximates (4.7) by a set of linear equations. Since

$$N(t) = \int_{\tau=0}^{t} \mathrm{d}N(\tau) \tag{4.8}$$

a final integration of the solution to (4.7) yields the wanted covariance function:

$$\mathrm{Cov}[N(t_1), N(t_2)] = \int_{\tau_1=0}^{t_1}\int_{\tau_2=0}^{t_2} \mathrm{Cov}[\mathrm{d}N(\tau_1), \mathrm{d}N(\tau_2)]. \tag{4.9}$$

As another example consider the relaxation problem for the case where the column $\mathscr{B}$ to time $t = 0$ is centrally prestressed to a normal force N_0 by a steel cable of cross-section area A_s and elasticity coefficient E_s. This problem may be solved by the same approximation (4.3) simply by setting

$$A\epsilon(t) = \frac{A}{A_sE_s}(N_0 - N(t)) + K(0, 0)N_0 = -\frac{A}{A_sE_s}\int_{\tau=0+}^{t} \mathrm{d}N(\tau) + K(0, 0)N_0 \text{ for } t > 0 \tag{4.10}$$

in (4.4). The eqns (4.5) and (4.7) are then valid with $K(t, \tau)$ changed to

$$K(t, \tau) + \frac{A}{A_sE_s}$$

and the r.h.s. of (4.5) put to $K(0, 0)N_0$. Then (4.5) may be written

$$\int_{\tau=0+}^{t} \left(K(t, \tau) + \frac{A}{A_sE_s}\right) \mathrm{d}E[N(\tau)] = (K(0, 0) - K(t, 0))N_0 \tag{4.12}$$

in which we have taken account of the jump $\mathrm{d}N(\tau) = N_0$ for $\tau = 0$.

It is worth-while to remark that direct use of the approximation (4.3) on the problems of Sections 2 and 3 leads to the right hand sides of the covariance relations derived there provided reversions of the stress field take place simultaneously everywhere within the same cross-section.

Example 3

Let the functions $K(t, T)$ and $c(t_1, t_2, T)$ be defined by, see (2.12)–(2.15),

$$K(t, T) = \left(1.5 \log\frac{t}{T} + 2\sqrt{\left(0.7 + \frac{4}{T}\right)}(1 - e^{-\alpha(t-T)}) + 3\right)\cdot 10^{-11}\,\mathrm{m^2/N},\ t \geq T$$

$$c(t_1, t_2, T) = \left(5 \log\frac{t_1}{T} + 12\sqrt{\left(0.7 + \frac{4}{T}\right)}\,e^{-\alpha(t_2-T)}(1 - e^{-\alpha(t_1-T)})\right)\cdot 10^{-18}\,\mathrm{m^5/N},\ t_2 \geq t_1 \geq T$$

where $\alpha = 1/30$ and time is in units of days. The constants in these expressions are not selected on basis of scientific inference from experimental data. They are merely crude engineering judgements of order of magnitudes.

A column with circular cross-section of diameter 0.15 m and length 1.5 m is subjected to a constant compression strain of 10^{-3} for $t \geq T = 28$ days. Numerical solution of the eqns

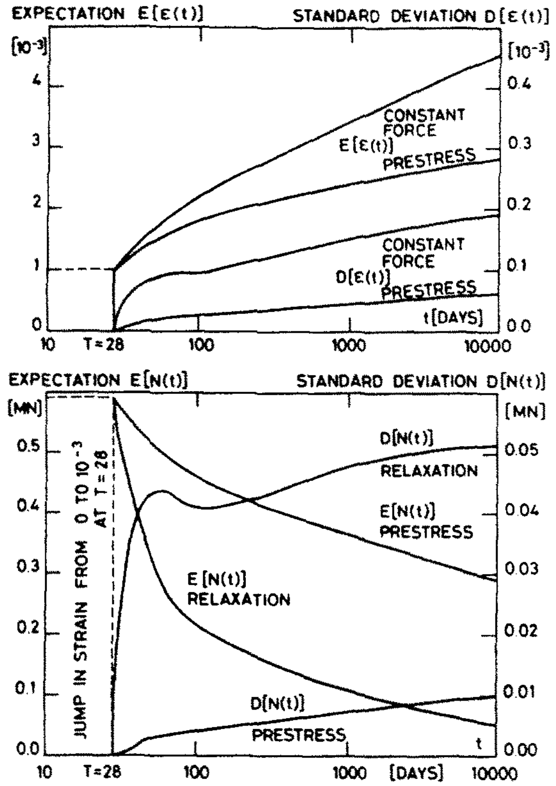

Fig. 3. Results related to the concrete column of Example 3. The top curves show the development in time of the expectation and standard deviation of the total concrete strain for a constant in time, externally applied normal force and for an internally applied steel cable prestressing force giving the same initial strain as the externally applied force.

The bottom curves show the expectation and standard deviation of the resulting normal force on the concrete section in the prestressing case and further in a relaxation case of constant in time total strain equal to the initial strain of the other cases.

(4.5)–(4.8) leads to the curves presented in Fig. 3 for $E[N(t)]$ and $D[N(t)]$ with label "relaxation". The "hump" on the standard deviation curve is an effect of the delayed elasticity. Figure 3 also shows curves for the same column subjected to a constant normal force for $t \geq T = 28$ days. The force has a value that gives the initial strain of 10^{-3} (for $t = T$) equal to the constant in time strain in the relaxation example. These curves are labeled "constant force" and they correspond to the formulae (2.10), (2.11) after multiplication with the relevant normal stress.

As a final example the same column is imagined to be centrally prestressed by a steel bar again to the same initial strain of 10^{-3}. The steel bar has circular cross-section of diameter 32 mm and the coefficient of elasticity is $E_s = 2 \cdot 10^{11}\,\mathrm{N/m^2}$. Numerical solution of (4.12) then leads to the curves for $E[N(t)]$ and correspondingly for $E[\epsilon(t)]$ both labeled "prestress". Numerical solution of the modification of (4.7) next leads to the standard deviation curves shown in Fig. 3 under label "prestress". Remark the considerable effect on the standard deviation $D[N(t)]$ of having the prestressing reinforcement as compared to the relaxation case of fixed prestrain.

5. SUMMARY AND CONCLUSIONS

With the purpose of calculating second moment representations of uncertain creep deformation quantities of beam-column type concrete structures an idealized stochastic process model of the creep strain behavior is suggested. The model is a second moment white noise process in geometrical space and stress space and a process with uncorrelated increments in the time dimension. This model is, in fact, an idealization which is a direct and natural consequence of earlier attempts [1, 2] to model concrete creep as a stochastic process phenomenon. The mathematical difficulties revealed herein are symptomatic for the subject. It turns out, however, that the second moment white noise model is just simple enough to make a great variety of problems within reach of solution by use of certain simplifying assumptions and approximations. With respect to the mean value functions of the creep quantities the model is consistent with usual visco-elasticity theory.

The main interest of application of the model is to get results in terms of internal forces when given the process model in terms of strains and stresses. Illustrations are worked out for a simple beam-column under compression and bending and also for a highly idealized reinforced beam subjected to bending.

Problems where the internal forces are dependent on the creep process itself are uttermost difficult (if not impossible) to deal with by exact mathematical methods. However, a certain linearization approximation opens up a solution procedure that makes it possible to solve general relaxation type problems.

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